INVARIANT TRANSFORMATION OF EULER EQUATIONS FOR PLANE STEADY FLOWS OF A PERFECT COMPRESSIBLE FLUID

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Transformation of Euler equations with initial invariant systems is presented. This permits the introduction of a new flow plane XY in which the fluid is subject to a different equation of state. The obtained relationship between flow parameters in the two planes makes it possible to derive (as in paper [1]) a solution of any problem in one of the planes, when a solution in the other plane is known.

1. Transformation of equations of motion. We shall consider the equations of motion and of continuity for a plane flow of a compressible fluid

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x},\qquad u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y},\qquad \frac{\partial}{\partial x}\rho u+\frac{\partial}{\partial y}\rho v=0 \qquad (1.1)$$

Using the continuity equation we may rewrite the first two equalities of this system in the divergent form

$$\frac{\partial}{\partial x}(p+\rho u^2)+\frac{\partial}{\partial y}\rho uv=0,\quad \frac{\partial}{\partial x}\rho uv+\frac{\partial}{\partial y}(p+\rho v^2)=0 \qquad (1.2)$$

It follows from (1.2) that we may introduce two independent functions X and Y by means of equalities

$$dX=(p+\rho v^2)dx-\rho uvdy,\qquad dY=-\rho uvdx+(p+\rho u^2)dy \qquad (1.3)$$

We introduce notations

$$\frac{u}{p}=U,\qquad \frac{v}{p}=V,\qquad \frac{\rho p}{p+\rho w^2}=\rho^\circ,\qquad -\frac{1}{p}=P \qquad (1.4)$$

Passing now in system (1.1) to the new variables X, Y we obtain

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{1}{\rho^\circ}\frac{\partial P}{\partial X},\quad U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{1}{\rho^\circ}\frac{\partial P}{\partial Y} \qquad (1.5)$$

$$\frac{\partial}{\partial X}\rho^\circ U+\frac{\partial}{\partial Y}\rho^\circ V=0\qquad \left(\frac{D(x,y)}{D(X,Y)}=p(p+\rho w^2)=+\frac{1}{P(P+\rho^\circ W^2)}\right)$$

Rejecting the case in which $p=0$ we conclude that variables X and Y will be independent. Thus, the flow of a fluid with parameters u, v, p and ρ in the xy-plane corresponds to the flow of another fluid in the XY-plane with parameters U, V, P, and ρ°, and the transition from the XY-plane to the physical plane is to be carried out in accordance with formulas ensuing from (1.3) and (1.4). Thus, by solving (1.3) with respect to dx and dy we obtain relationships $x=x(X,Y)$, $y=y(X,Y)$ which are defined by curvilinear integrals taken along any path connecting two given points of plane

$$x = -\int_L (P + \rho^\circ V^2)\,dX - \rho^\circ UV\,dY \quad y = -\int_L -\rho^\circ UV dX + (P + \rho^\circ U^2)\,dY \tag{1.6}$$

2. Streamlines. From the appropriate continuity equations of (1.1) and (1.5) we introduce functions of the ψ and Ψ type

$$\frac{\partial\psi}{\partial x} = -\rho v, \quad \frac{\partial\psi}{\partial y} = \rho u; \qquad \frac{\partial\Psi}{\partial X} = -\rho^\circ V, \quad \frac{\Psi}{\partial Y} = \rho^\circ U \tag{2.1}$$

We select in the xy-plane a streamline ψ = const, and shall find its image in the XY-plane. Since

$$d\psi = -\rho v dx + \rho u dy \tag{2.2}$$

hence Formulas (1.3) may be written as

$$dX = pdx - vd\psi, \qquad dY = pdy + ud\psi \tag{2.3}$$

With the aid of (2.2) we eliminate dx and dy from (2.3), and obtain

$$d\psi = -\frac{\rho v}{p + \rho w^2}\,dX + \frac{\rho u}{p + \rho w^2}\,dY \tag{2.4}$$

Taking into account (1.4) we derive Formula

$$d\psi = -\rho_0 V dX + \rho_0 U dY = d\Psi \tag{2.5}$$

Hence in this transformation line ψ = const becomes line Ψ = const, $\psi = \Psi$, and the angles of inclination of tangents to streamlines $\Psi = c$ and $\Psi = c$ to their corresponding axes are equal.

Thus, the transformations obtainable with the aid of Formulas (1.3) convert streamlines inthe xy-plane into streamlines in plane XY. In the case of an irrotational flow this transformation retains the orthogonality of streamlines and of equipotential lines. A fictitious velocity potential may be introduce in the XY-plane by means of relation

$$d\varphi = UdX + VdY \tag{2.6}$$

It follows from (2.5) and (2.6) that the hodograph transformation would yield the already available equations in which the dependence of density and velocity on pressure will be different.

3. Equations of state. In the physical plane the following equality is fulfilled along the streamlines:

$$\rho w dw + dp = 0 \tag{3.1}$$

A similar relationship will also obtain in the XY-plane, and

$$\rho^\circ W dW + dP = \frac{1}{J}(\rho w dw + dp) \tag{3.2}$$

Here J is the Jacobean of transformation.

The relationship between density and pressure can be determined from (1.4), (3.1) and (3.2).

We shall consider the flow of an incompressible fluid in the XY-plane. From (1.4) and (3.1) follows that in this case the flow corresponds to the flow of a Chaplygin hypothetical gas

$$p = c\left(1 + \frac{\rho_1}{\rho}\right) \tag{3.3}$$

Having determined U and V from expressions (2.5) for Ψ, and P from (3.2), we readily find the corresponding solution for a compressible fluid by a recalculation of Formulas (1.6).

When considering the problem of flow of a compressible fluid stream past a circle we go over to the physical plane, we obtain a flow past an oval profile symmetric relative to the x-axis.

Moving along streamline ψ = const we obtain from (2.3)

$$dX = pdx, \qquad dY = pdy \tag{3.4}$$

When a streamline is a line of constant pressure which occurs at the free surface, then from (3.4) follows that

$$X = p_\infty x + c, \qquad Y = p_\infty y + c_1$$

Hence the free surface form is readily found whenever its image is known in one of the planes.

BIBLIOGRAPHY

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